Trilateration using four Hydrophones and measured time delay

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For additional notes see Brian Moore btmoore44@gmail.com

Hydrophones h_0 h_X h_Y and h_Z detect a signal from a sound-pulse source, and compare the time differences between them to find the exact position of the device relative to h_0.

Ping source is at location (x,y,z)

h_0 is at location (0,0,0) (the origin)
h_x is at location (δ,0,0)
h_y is at location (0,ε,0)
h_z is at location (0,0,ζ)

When a signal is detected, each hydrophone creates a time stamp, t_0 t_X t_Y and t_Z. t_0 will act as the common reference at the origin. Thus the time differences measured are:

$\Delta t_X = t_0 - t_X$, $\Delta t_Y = t_0 - t_Y$ $\Delta t_Z = t_0 - t_Z$

The three time-difference measurements are multiplied by the speed of sound (c_s) to determine the difference in distance from the signal source for each hydrophone.

$\Delta t_X * c_s = \Delta x$, $\Delta t_Y * c_s = \Delta y$ $\Delta t_Z * c_s = \Delta z$

Δx, Δy, and Δz will be the measured values used in the calculation. Thus the final derivation should be in terms of Δx, Δy, and Δz, and any derivative values of the three.

ρ0 indicates the absolute difference from h_0 to the signal source at (x,y,z)

$\rho 0 = \text{sqrt}(x^2 + y^2 + z^2)$

ρx, ρy, and ρz indicate the distance from their respective hydrophones to the source:

$\rho x = \text{sqrt}((x-\delta)^2 + y^2 + z^2)$, $\rho y = \text{sqrt}(x^2 +(y-\varepsilon)^2 + z^2)$, $\rho z = \text{sqrt}(x^2 + y^2 +(z-\zeta)^2)$

The differences in distance then, each are the difference between ρ0 and the distance from the hydrophone's location to the signal source.

$\Delta x = \rho 0 - \rho x = \text{sqrt}(x^2 + y^2 + z^2) - \text{sqrt}((x-\delta)^2 + y^2 + z^2)$
$\Delta y = \rho 0 - \rho y = \text{sqrt}(x^2 + y^2 + z^2) - \text{sqrt}(x^2 +(y-\varepsilon)^2 + z^2)$
$\Delta z = \rho 0 - \rho z = \text{sqrt}(x^2 + y^2 + z^2) - \text{sqrt}(x^2 + y^2 +(z-\zeta)^2)$

With these three known comparisons, Δx, Δy, Δz, and the three known hydrophone locations δ, ε, and ζ along their respective axes, we can solve for exact values of x, y, and z – thus locating the signal source.

Using h_X and its measurements we will derive a formula for x, which will be identical to the derivations for y and z with their respective measurements.

$$\Delta x = \rho 0 - \rho x \tag{1}$$

$$\Delta x = \mathrm{sqrt}(x^2 + y^2 + z^2) - \mathrm{sqrt}((x-\delta)^2 + y^2 + z^2) \tag{2}$$

$$\rho 0^2 - \rho x^2 = (x^2 + y^2 + z^2) - ((x-\delta)^2 + y^2 + z^2)$$

$$\rho 0^2 - \rho x^2 = x^2 - (x-\delta)^2$$

$$\rho 0^2 - \rho x^2 = x^2 - x^2 + 2\delta x - \delta^2$$

$$\rho 0^2 - \rho x^2 = 2\delta x - \delta^2 \tag{3}$$

$$\rho 0^2 - \rho x^2 = (\rho 0 + \rho x) * (\rho 0 - \rho x) \tag{4}$$

$$\rho 0 - \rho x = (2\delta x - \delta^2) / (\rho 0 + \rho x) \tag{5}$$

$$\rho x = \rho 0 - \Delta x \tag{6}$$

$$\Delta x = (2\delta x - \delta^2) / (2\rho 0 - \Delta x) \tag{7}$$

$$2 \rho 0\Delta x - \Delta x^2 = 2\delta x - \delta^2$$

$$2\delta x = 2\rho 0\Delta x - \Delta x^2 + \delta^2$$

$$x = (- \Delta x^2 + 2 \rho 0\Delta x + \delta^2) / (2\delta) \tag{8}$$

We can duplicate (1) through (8) to derive similar expressions for y and z.

$$y = (- \Delta y^2 + 2 \rho 0\Delta y + \varepsilon^2) / (2\varepsilon) \tag{9}$$

$$z = (- \Delta z^2 + 2 \rho 0\Delta z + \zeta^2) / (2\zeta) \tag{10}$$

We discover that x, y, and z do not depend solely on the difference in distance measured at their own hydrophones, but also on ρ0, the absolute distance away from the ping source, which is itself a magnitude of the combination of coordinates x,y, and z. In other words, they are inter-related (as would be expected).

It therefore becomes necessary to find an expression for ρ0 in terms of Δx, Δy, Δz, δ, ε, and ζ. Luckily, $\rho 0^2$ is a function of x^2, y^2, and z^2, and we have an equation relating x to ρ0 as well as similar equations for y and z.

$$x^2 = [(-\Delta x^2 + 2\ \rho 0 \Delta x + \delta^2) / (2\delta)]^2 \tag{11}$$

$$x^2 = [(\Delta x^4 - 2\ \rho 0 \Delta x^3 - \Delta x^2\ \delta^2) + (-2\ \rho 0 \Delta x^3 + 4\rho 0^2 \Delta x^2 + 2\rho 0 \Delta x\ \delta^2) + (-\Delta x^2\ \delta^2 + 2\rho 0 \Delta x\ \delta^2 + \delta^4)] / (4\delta^2)$$

$$x^2 = (4\rho 0^2 \Delta x^2 - 4\ \rho 0 \Delta x^3 + 2\rho 0 \Delta x\ \delta^2 + \Delta x^4 - 2\ \Delta x^2\ \delta^2 + \delta^4) / (4\delta^2)$$

$$x^2 = \rho 0^2 * [4\Delta x^2 / 4\delta^2] + \rho 0 * [(-4\Delta x^3 + 4\Delta x\ \delta^2) / 4\delta^2] + [(\Delta x^2 - \delta^2)^2 / 4\delta^2]$$

$$x^2 = \rho 0^2 * [(\Delta x/ \delta)^2] + \rho 0 * [(\Delta x/\delta^2)* (\delta^2 - \Delta x^2)] + [((\Delta x^2 - \delta^2) / 2\delta)^2] \tag{12}$$

Generating similar results for y and z provide:

$$y^2 = \rho 0^2 * [(\Delta y/ \varepsilon)^2] + \rho 0 * [(\Delta y/\varepsilon^2)* (\varepsilon^2 - \Delta y^2)] + [((\Delta y^2 - \varepsilon^2) / 2\varepsilon)^2] \tag{13}$$

$$z^2 = \rho 0^2 * [(\Delta z/ \zeta)^2] + \rho 0 * [(\Delta z/\zeta^2)* (\zeta^2 - \Delta z^2)] + [((\Delta z^2 -\zeta^2) / 2\zeta)^2] \tag{14}$$

Then we can combine (12), (13), and (14) to solve for ρ0 entirely in terms of measured values.

$$x^2 = \rho 0^2 * [(\Delta x/ \delta)^2] + \rho 0 * [(\Delta x/\delta^2)* (\delta^2 - \Delta x^2)] + [((\Delta x^2 - \delta^2) / 2\delta)^2]$$

$$x^2 = \rho 0^2 * [xa] + \rho 0 * [xb] + [xc]$$

$$xa = (\Delta x/ \delta)^2 , \quad xb = (\Delta x/\delta^2)* (\delta^2 - \Delta x^2), \quad xc = ((\Delta x^2 - \delta^2) / 2\delta)^2 \tag{15}$$

$$ya = (\Delta y/ \varepsilon)^2, \quad yb = (\Delta y/\varepsilon^2)* (\varepsilon^2 - \Delta y^2), \quad yc = ((\Delta y^2 - \varepsilon^2) / 2\varepsilon)^2 \tag{16}$$

$$za = (\Delta z/ \zeta)^2, \quad zb = (\Delta z/\zeta^2)* (\zeta^2 - \Delta z^2), \quad zc = ((\Delta z^2 -\zeta^2) / 2\zeta)^2 \tag{17}$$

$$\rho 0^2 = x^2 + y^2 + z^2$$

$$\rho 0^2 = \rho 0^2 * [xa + ya + za] + \rho 0 * [xb + yb + zb] + [xc + yc + zc]$$

$$0 = \rho 0^2 * [xa + ya + za - 1] + \rho 0 * [xb + yb + zb] + xc + yc + zc \tag{18}$$

With Equation 18 we get a representation of $\rho 0^2$ in a quadratic format, with coefficients that are all functions of known values. Solving the quadratic equation yields two values of ρ0. Selecting the appropriate value (not covered here) then allows you to plug in the value of ρ0 into equations (8) (9), and (10). Evaluating those equations yields the values of x, y, and z.

Three hydrophones and a depth sensor:

In the current arrangement, a limit of 3 hydrophones requires the use of a depth sensor to supplement the missing information. By measuring the absolute depth of the sub, and knowing the fixed depth of the ping source, the value z is their difference. As seen below, the equation is easily modified to accommodate a lack of a time-difference measurement along an axis, in exchange for knowing the absolute value of the coordinate.

$$\rho 0^2 = x^2 + y^2 + z^2$$

$$\rho 0^2 = \rho 0^2 * [xa + ya] + \rho 0 * [xb+yb] + [xc + yc] + z^2$$

$$0 = \rho 0^2 * [xa + ya - 1] + \rho 0 * [xb+yb] + [xc + yc] + z^2 \qquad (19)$$

As equation 19 shows, the $\rho 0^2$ and $\rho 0$ coefficients decrease while the constant (significantly) increases, balancing out the quadratic equation and outputting the same values when evaluated.

Additional notes:

It is worth noting that Δx, Δy and Δz all have a maximum value of δ ε and ζ respectively; when the source exists straight along the x, y or z axis. As the values of xa, xb, and xc are all composed of functions of Δx divided by at least the same order of δ, or differences of δ and Δx, all absolute values of xa, xb, and xc are going to be less than 1. y and z follow the same pattern, meaning that the coefficients for the quadratic equation are all the sum of three values less than one. Thus, a, b, and c will all be relatively small.

Additionally the values of Δx, Δy and Δz are inter-related and geometrically constrained. An increase in one value can only come at the cost of a decrease in another value, and vice versa. For instance, you could not have a ping source that exists directly along two axes, so you can never have $\Delta x=\delta$ and $\Delta y=\varepsilon$ simultaneously.

The exception to this is found when x, y, and z are all between 0 and δ, ε, and ζ respectively. If the ping source exists at the origin (0,0,0), ie the location of hydrophone h_0, then $\Delta x=-\delta$, $\Delta y=-\varepsilon$, and $\Delta z=-\zeta$ simultaneously. Likewise, if the ping source was at the centroid of the tetrahedron generated by the four hydrophones, then $\Delta x = \Delta y = \Delta z = 0$.

However, with the physical constraints of the system, the ping source will never fall within the tetrahedron, permitting potentially useful exploitation of these mathematical relationships without concern for the exceptional cases.