Pinger Bearing

This page describes the math used to calculate our bearing towards the pinger using our hydrophones. A solution is derived in 2 dimensions to show how it works, then the final result from that is extrapolated to 3 dimensions. The speed of sound of water is defined as c_s.

Assumptions

Planar Wave

Imagine dropping a rock in the middle of a pond. At first the ripples look very circular, but by the time they reach the shore they look like straight lines. We assume that the wave front of the ping is planar, which basically means that even though the wave front is curved, the curvature is small enough from the perspective of the hydrophones such that the wave front actually looks flat. This makes our calculations a lot easier.
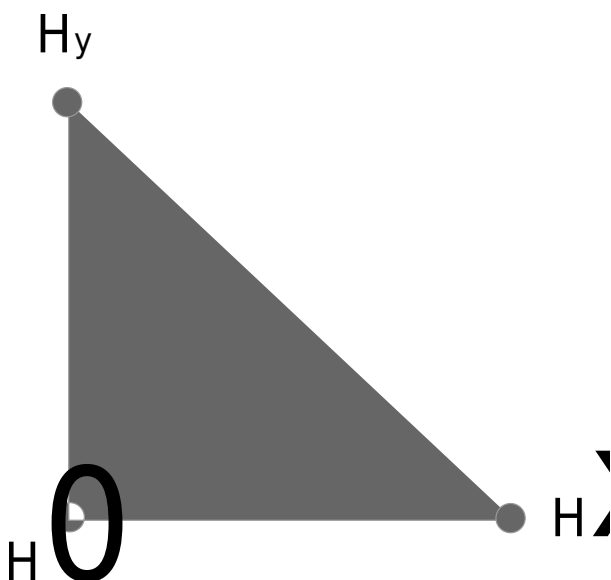
2D Derivation

We have 3 hydrophones, located at the following positions:
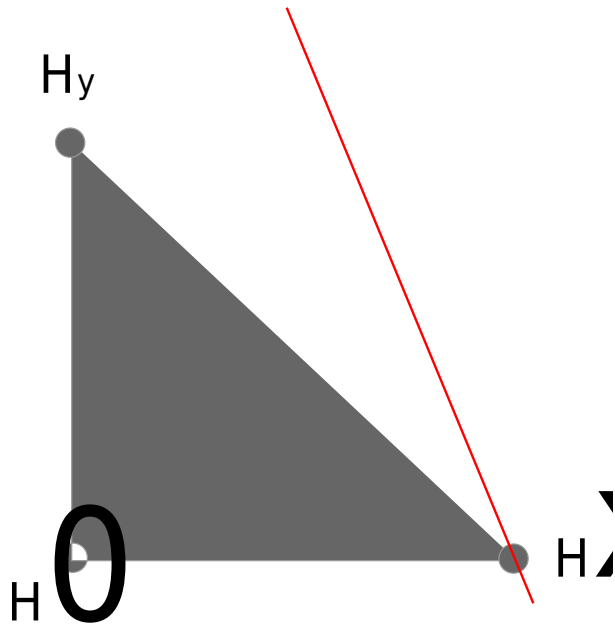
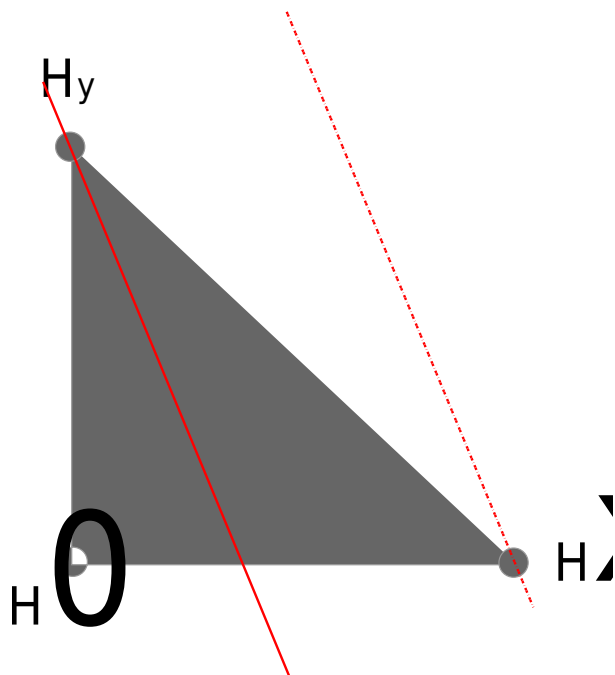H_0 is at location $(0,0)$
H_x is at location $(x,0)$
H_y is at location $(0,y)$
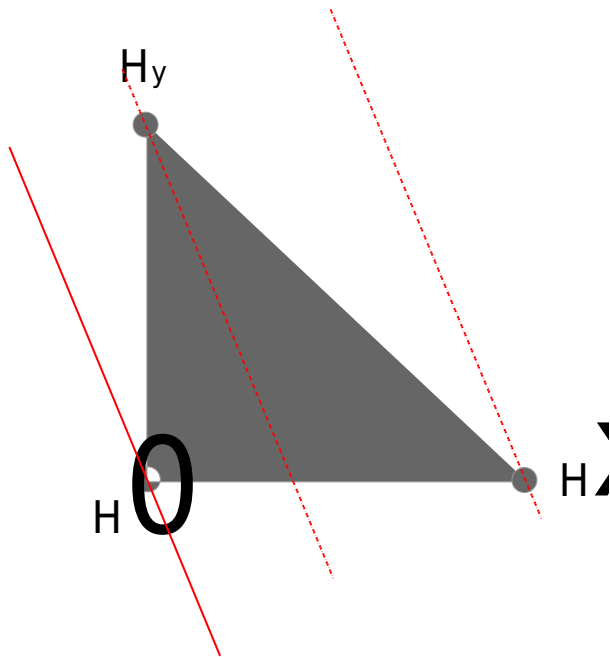

We shall simulate a wave traveling towards our hydrophone array. First, it hits H_x at time t_x

H_y

H_0

H_x

The wave continues, hitting H_y at time t_y

H_y

H_0

H_x

Finally, the wave hits H_0 at time t_0

H_y

H O

H_x

Because the wave emanating from the pinger hit H_x before it hit H_0, we know that H_x must be closer to the pinger than H_0. In fact, H_x must be exactly $d_x = c_s(t_0 - t_x)$ meters closer to the pinger than H_0. Likewise, H_y is $d_y = c_s(t_0 - t_y)$ meters closer to the pinger

H_y

H O d_y d_x H_x
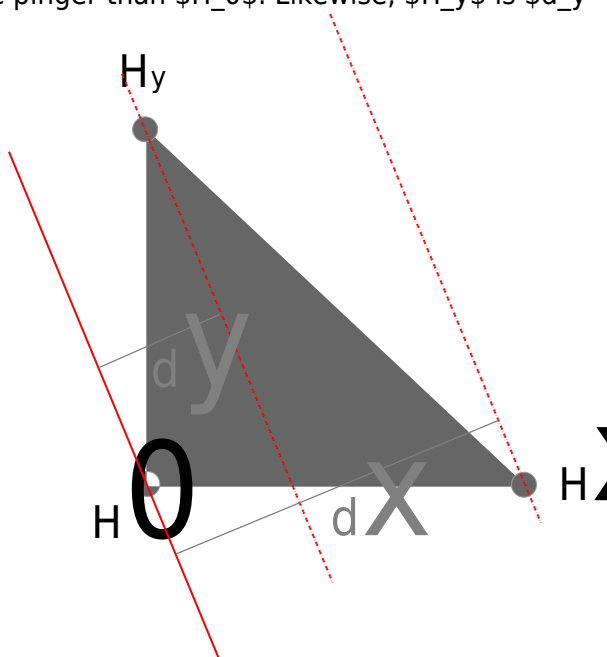
than H_0.

Our main goal is to calculate a vector pointing from the H_0 to the pinger. This vector is perpendicular to the wave front, and is shown in blue in the diagram below:
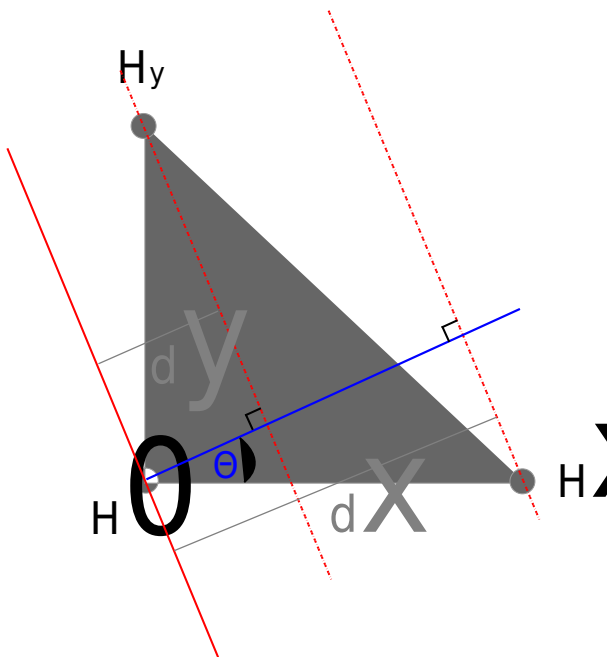
We can also describe the vector in i,j notation: $$ \begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} $$

Also recall the cosine identity: $$ \cos(\theta) = \frac{\text{adj}}{\text{hyp}} $$

Notice how the wavefront, axis to the hydrophones, and our vector make a few right triangles? We
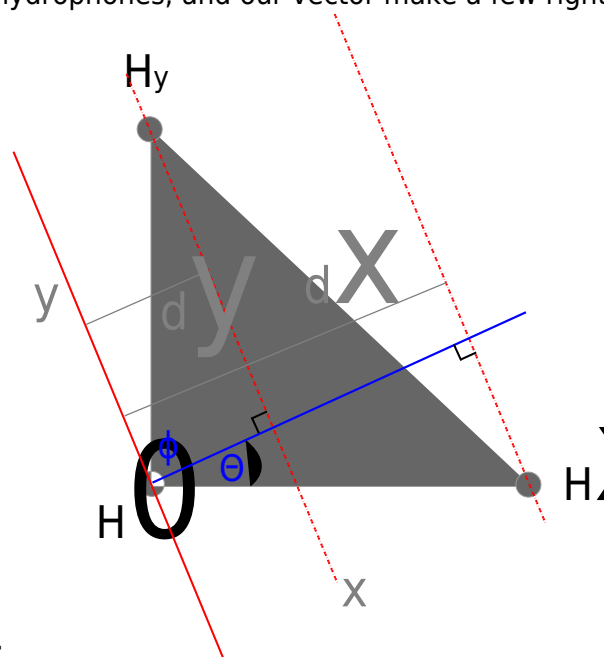
highlighted two in green and magenta:

Let's focus on the green triangle first. We know the hypotenuse of the triangle is x, and the long side of the triangle is d_x, therefore, we know $\cos(\theta)$: $$ \cos(\theta) = \frac{d_x}{x} $$

Next, let's look at the magenta triangle. We know the hypotenuse is y and the short side is d_y, and can calculate: $$ \cos(\phi) = \frac{d_y}{y} $$

However, recall that $\phi = 90-\theta$, therefore $$ \cos(\phi) = \cos(90-\theta) = \sin(\theta) $$

We now know how to calculate $\hat{i}$ and $\hat{j}$: $$ \begin{bmatrix} \hat{i} \\ \hat{j}

$$ \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \begin{bmatrix} d_x/x \\ d_y/y \end{bmatrix} $$

3D Result

Extrapolating the 2D result into 3 dimensions, we have:

H_0 is at location $(0,0,0)$
H_x is at location $(x,0,0)$
H_y is at location $(0,y,0)$
H_z is at location $(0,0,z)$

Given that our hydrophones receive the ping at times t_0, t_x, t_y, and t_z respectively, we can calculate:

$$ \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = c_s \begin{bmatrix} t_0 - t_x \\ t_0 - t_y \\ t_0 - t_z \end{bmatrix} $$

$$ \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} d_x/x \\ d_y/y \\ d_z/z \end{bmatrix} $$

Generalized Case

Below describes how to calculate the bearing in the event that the three hydrophones are not orthogonal to each other or do not lie along the x, y, or z axes. This solution still requires the far-field assumption be valid. The derivation is covered first. The direct solution can be found further below.

Generalized Derivation

The measurement of static vector fields by 3 or more sensors is a very common application. Magnetometers, accelerometers, depth sensor arrays, and hydrophones on this submarine can all potentially be analyzed in this way. A measurement typically manifests as the dot product between a vector describing the particular sensor, and the vector describing the phenomenon being measured.

A dot product is the sum of the products of each corresponding component of two vectors. This scalar sum is equal to the product of the magnitudes of the vectors, multiplied by the cosine of the angle between them.

$$ \vec{a} = (a_x, a_y, a_z) \\ mag(\vec{a}) = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \\ \vec{a} \cdot \vec{b} = a_xb_x + a_yb_y + a_zb_z = |\vec{a}||\vec{b}|\cos(\theta) $$

In this case, the vectors are the position vector of a given hydrophone H_n and the position vector of the pinger P.

$$\vec{H_n} \cdot \vec{P} = |\vec{H_n}||\vec{P}|\cos(\theta)$$

However, because we're calculating a difference between our sensor and a reference, the measurement does not actually reflect a dot product of these two vectors, but rather a dot product between the hydrophone vector H_n and the unit vector of the pinger $\hat{P}$. This is also known as a 'projection.' Phrased another way, it describes how large of a component in H_n lies parallel to $\vec{P}$. This is fine, because we really aren't looking to calculate the exact location of the pinger, but simply the direction to it.

Recall that a unit vector is a vector divided by its own magnitude. The resulting vector is parallel to the original, but has a magnitude of 1. The unit vector $\hat{P}$ is the directional vector $(\hat{i},\hat{j},\hat{k})$ we're after.

$$ \hat{P} = \frac{\vec{P}}{|\vec{P}|} = \frac{(P_x,P_y,P_z)}{|\vec{P}|} = (\hat{i},\hat{j},\hat{k}) \\ |\hat{P}| = 1\\ proj_\vec{P}\vec{H_n} = \vec{H_n} \cdot \hat{P} = \frac{|\vec{H_n}||\vec{P}|}{|\vec{P}|}\cos(\theta) = |\vec{H_n}|\cos(\theta) = d_n \\ $$

In the trivial case where our hydrophone lies at position $H_x = (x,0,0)$ along the x-axis, we get the same result as our 2D derivation, as the x-coordinate x is also the vector's magnitude:

$$ (x,0,0) \cdot \hat{P} = |x|\cos{\theta} = \sqrt{x^2 + 0^2 + 0^2}\cos{\theta} = x\cos{\theta} = d_x $$

Recall that Matrix multiplication consists of systematically taking dot products.

$$ H_n \cdot \hat{P} = H_n \hat{P}^T = \begin{bmatrix} H_{nx} & H_{ny} & H_{nz} \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \\ \end{bmatrix} = \begin{bmatrix} d_n \end{bmatrix} $$

Generalized Solution

Thus we can organize systems of dot products and measurements as the result of a single matrix multiplication.

$$ H \hat{P}^T = D \\ \begin{bmatrix} [& \vec{H_1} &] \\ [& \vec{H_2} &] \\ [& \vec{H_3} &] \\ \end{bmatrix} \begin{bmatrix} \hat{P_x} \\ \hat{P_y} \\ \hat{P_z} \\ \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \\ \begin{bmatrix} H_{1x} & H_{1y} & H_{1z} \\ H_{2x} & H_{2y} & H_{2z} \\ H_{3x} & H_{3y} & H_{3z} \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \\ \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} $$

To solve for our bearing $\hat{P}$, we simply solve the matrix equation. Multiplying both sides by the inverse of the hydrophone location matrix H provides the solution. This is the fully generalized result for hydrophone measurements in the far-field.

$$ H^{-1}H\hat{P}^T = H^{-1}D \\ \hat{P}^T = H^{-1}D \\ \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \\ \end{bmatrix} = \begin{bmatrix} H_{1x} & H_{1y} & H_{1z} \\ H_{2x} & H_{2y} & H_{2z} \\ H_{3x} & H_{3y} & H_{3z} \end{bmatrix}^{-1} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} $$

We can see based on this structure that we need at least three hydrophones measurements (d_1, d_2, and d_3) to solve for the three unknown elements of unit vector $\hat{P}$. Furthermore, the three position vectors of the hydrophones ($\vec{H_1}$, $\vec{H_2}$, and $\vec{H_3}$) must

constitute an invertible, full-rank matrix. In this case that simply means the three hydrophones cannot all lie on the same plane.

As a sanity-check, this formulation should re-produce the degenerate case where all hydrophones lie perfectly along the x, y, and z axes.

$$ H_{xyz} \hat{P}^T = D_{xyz} \\ \begin{bmatrix} [& \vec{H_x} &] \\ [& \vec{H_y} &] \\ [& \vec{H_z} &] \\ \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \\ \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \\ \end{bmatrix} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \\ \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \\ \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}^{-1} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \\ \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \\ \end{bmatrix} = \begin{bmatrix} d_x/x \\ d_y/y \\ d_z/z \end{bmatrix} $$