Trilateration

Below is the math for calculating the location of the pinger in the water relative to our submarine. Because we know the absolute location of the pinger in the pool, we can calculate the sub's position in the pool.

Original derivation by Brian Moore.

Problem Setup

The derivation is based on the assumption that we have 4 hydrophones. One is considered the reference hydrophone, while the others are located along the x, y, and z axes.

h_0 is at location $(0,0,0)$
h_x is at location $(\delta,0,0)$
h_y is at location $(0,\epsilon,0)$
h_z is at location $(0,0,\zeta)$

We define the ping source is at location $$ l_{pinger} = (x,y,z) $$

When a ping is received by the hydrophones, the hardware outputs delta-timestamps Δt_x, Δt_y, Δt_z, which corresponds to the difference in time between when the ping was received by h_0 and $h_{x,y,z}$, respectively.

Let's define p_0 as the absolute distance between h_0 and the pinger at location (x,y,z). $$ p_0 = \sqrt{x^2 + y^2 + z^2} $$

These three time differences are multiplied by the speed of sound in water (c_s) to determine the difference in distance between the the reference hydrophone and pinger and each other hydrophone $h_{x,y,z}$. $$ \Delta x = \Delta t_x * c_s \\ \Delta y = \Delta t_y * c_s \\ \Delta z = \Delta t_z * c_s \\ $$ In other words, h_x is Δx meters farther from the pinger than h_0, and h_0 is p_0 meters from the pinger.

The final calculations for x, y, and z will be in terms of Δx, Δy, and Δz

Let's define the distances from the other hydrophones to the pinger: $$ p_x = \sqrt{(x-\delta)^2 + y^2 + z^2} \\ p_y = \sqrt{x^2 + (y-\epsilon)^2 + z^2} \\ p_z = \sqrt{x^2 + y^2 + (z-\zeta)^2} \\ $$

Let's put this information together: $$ \Delta x = p_0 - p_x = \sqrt{x^2 + y^z + z^2} - \sqrt{(x-\delta)^2 + y^2 + z^2} \\ \Delta y = p_0 - p_y = \sqrt{x^2 + y^z + z^2} - \sqrt{x^2 + (y-\epsilon)^2 + z^2} \\ \Delta z = p_0 - p_z = \sqrt{x^2 + y^z + z^2} - \sqrt{x^2 + y^2 + (z-\zeta)^2} \\ $$

Since we know Δx, Δy, Δz, δ, ϵ, and ζ we can solve for x, y, z!

Solving for Position

Using h_x and its measurements, we will derive a formula for x. The same steps can be used for solving y and z.

First, let's start with the final equation from the previous section: $$ \Delta x = \sqrt{x^2 + y^z + z^2} - \sqrt{(x-\delta)^2 + y^2 + z^2} $$ Also recall that $$ p_0 = \sqrt{x^2 + y^2 + z^2} \\ p_x = \sqrt{(x-\delta)^2 + y^2 + z^2} $$ Therefore $$ p_0^2 - p_x^2 = (x^2 + y^2 + z^2) - ((x-\delta)^2 + y^2 + z^2) $$ Simplifying, we get $$ p_0^2 - p_x^2 = x^2 - (x-\delta)^2 \\ = x^2 - (x^2 - 2\delta x + \delta^2) \\ = 2 \delta x - \delta^2 $$ Note that $$ p_0^2 - p_x^2 = (p_0+p_x)(p_0-p_x) \\ p_0-p_x = \frac{p_0^2 - p_x^2}{p_0+p_x} $$ Therefore $$ p_0-p_x = \frac{2 \delta x - \delta^2}{p_0+p_x} $$

Recall that $$ \Delta x = p_0 - p_x \\ p_x = p_0 - \Delta x $$

Combining this with the previous equation, we can substitute Δx for p_0-p_x on the left side and $p_0 - \Delta x$ for p_x on the right side, which gives us $$ \Delta x = \frac{2 \delta x - \delta^2}{2p_0 - \Delta x} $$ Let's now take this result and solve for x $$ \Delta x = \frac{2 \delta x - \delta^2}{2p_0 - \Delta x} \\ \Delta x (2p_0 - \Delta x) = 2 \delta x - \delta^2 \\ \Delta x (2p_0 - \Delta x) + \delta^2 = 2 \delta x \\ \frac{\Delta x (2p_0 - \Delta x) + \delta^2}{2 \delta} = x \\ $$ Following this same technique, we can also derive equations for y and z. $$ x = \frac{\Delta x (2p_0 - \Delta x) + \delta^2}{2 \delta} \\ y = \frac{\Delta y (2p_0 - \Delta y) + \epsilon^2}{2 \epsilon} \\ z = \frac{\Delta z (2p_0 - \Delta z) + \zeta^2}{2 \zeta} $$ It can be seen that x, y, and z are dependent not only in the time delays, but also on p_0. Therefore, it is necessary to find an expression for p_0 in terms of Δx, Δy, Δz, δ, ϵ, and ζ.

First, remember that $$ p_0 = \sqrt{x^2 + y^2 + z^2} \\ p_0^2 = x^2 + y^2 + z^2 $$ We can solve for p_0^2 using x^2, y^2, and z^2. Let's take our result for x, calculate x^2, and expand. $$ x = \frac{\Delta x (2p_0 - \Delta x) + \delta^2}{2 \delta} \\ x^2 = \left(\frac{\Delta x (2p_0 - \Delta x) + \delta^2}{2 \delta}\right)^2 \\ x^2 = \frac{(\Delta x (2p_0 - \Delta x) + \delta^2)^2}{(2 \delta)^2} \\ x^2 = \frac{(\delta^2 + 2p_0\Delta x - \Delta x^2)^2}{(2 \delta)^2} \\ x^2 = \frac{(-\delta^2 \Delta x^2 - 2p_0\Delta x^3 + \Delta x^4) + (2p_0 \delta^2 \Delta x + 4p_0^2 \Delta x^2 - 2p_0 \Delta x^3) + (\delta^4 + 2p_0 \delta^2 \Delta x - \delta^2 \Delta x^2)}{4 \delta^2} \\ x^2 = \frac{4p_0^2 \Delta x^2 - 4p_0 \Delta x^3 + 4p_0 \delta^2 \Delta x + \Delta x^4 - 2\delta^2\Delta x^2 + \delta^4 } {4 \delta^2} \\ x^2 = \frac{p_0^2 (4\Delta x^2) + p_0 (-4\Delta x^3 + 4 \delta^2 \Delta x) + (\Delta x^2 - \delta^2)^2} {4 \delta^2} \\ x^2 = p_0^2 \frac{4\Delta x^2}{4 \delta^2} + p_0 \frac{-4\Delta x^3 + 4 \delta^2 \Delta x}{4 \delta^2} + \frac{(\Delta x^2 - \delta^2)^2}{4 \delta^2} \\ x^2 = p_0^2 \frac{4\Delta x^2}{4 \delta^2} + p_0 \frac{4\Delta x}{4\delta^2}(-\Delta x^2 + \delta^2) + \frac{(\Delta x^2 - \delta^2)^2}{4 \delta^2} \\ x^2 = p_0^2 \left(\frac{\Delta x}{\delta}\right)^2 + p_0 \frac{\Delta x}{\delta^2}(\delta^2 +\Delta x^2) + \left(\frac{\Delta x^2 - \delta^2}{2 \delta}\right)^2 \\ $$

We can derive a similar equation for y^2 and z^2 $$ x^2 = p_0^2 \left(\frac{\Delta x}{\delta}\right)^2 + p_0 \frac{\Delta x}{\delta^2}(\delta^2 +\Delta x^2) + \left(\frac{\Delta x^2 - \delta^2}{2 \delta}\right)^2 \\ y^2 = p_0^2 \left(\frac{\Delta y}{\epsilon}\right)^2 + p_0 \frac{\Delta y}{\epsilon^2}(\epsilon^2 +\Delta y^2) + \left(\frac{\Delta y^2 - \epsilon^2}{2 \epsilon}\right)^2 \\ z^2 = p_0^2 \left(\frac{\Delta z}{\zeta}\right)^2 + p_0 \frac{\Delta z}{\zeta^2}(\zeta^2 +\Delta z^2) + \left(\frac{\Delta z^2 - \zeta^2}{2 \zeta}\right)^2 \\ $$

Notice that each of these equations are in the form $$ p_0^2 e + p_0 f + g $$

Let's define the following variables: $$ a_x = \left(\frac{\Delta x}{\delta}\right)^2 \qquad b_x = \frac{\Delta x}{\delta^2}(\delta^2 +\Delta x^2) \qquad c_x = \left(\frac{\Delta x^2 - \delta^2}{2 \delta}\right)^2 \\ a_y = \left(\frac{\Delta y}{\epsilon}\right)^2 \qquad b_y = \frac{\Delta y}{\epsilon^2}(\epsilon^2 +\Delta y^2) \qquad c_y = \left(\frac{\Delta y^2 - \epsilon^2}{2 \epsilon}\right)^2 \\ a_z = \left(\frac{\Delta z}{\zeta}\right)^2 \qquad b_z = \frac{\Delta z}{\zeta^2}(\zeta^2 +\Delta z^2) \qquad c_z = \left(\frac{\Delta z^2 - \zeta^2}{2 \zeta}\right)^2 $$

Therefore, we can rewrite x^2, y^2, and z^2 as $$ \begin{equation} \label{eq:squared_pos_from_p0_simplified} x^2 = p_0^2 a_x + p_0 b_x + c_x \\ y^2 = p_0^2 a_y + p_0 b_y + c_y \\ z^2 = p_0^2 a_z + p_0 b_z + c_z \end{equation} $$ Recall that $$ \begin{equation} \label{eq:p_0_from_position} p_0^2 = x^2 + y^z + z^2 \end{equation} $$ Combining equations $\ref{eq:squared_pos_from_p0_simplified}$ and $\ref{eq:p_0_from_position}$, we get $$ p_0^2 = (p_0^2 a_x + p_0 b_x + c_x) + (p_0^2 a_y + p_0 b_y + c_y) + (p_0^2 a_z + p_0 b_z + c_z) \\ p_0^2 = p_0^2(a_x + a_y + a_z) + p_0(b_x + b_y + b_z) + (c_x+c_y+c_z) \\ 0 = p_0^2(a_x + a_y + a_z - 1) + p_0(b_x + b_y + b_z) + (c_x+c_y+c_z) $$

Letting $$ a = a_x + a_y + a_z - 1 \\ b = b_x + b_y + b_z \\ c = c_x + c_y + c_z $$ we can simplify to $$ \begin{equation} \label{eq:p_0_final} 0 = p_0^2 a + p_0 b + c \end{equation} $$

Equation $\ref{eq:p_0_final}$ is a simple quadratic equation we can use to solve for p_0, which will give us 2 possible values. Now that we have p_0, we can plug it in and solve for x,y, and z!

Error Analysis

Additional Notes