Kalman Filter Introduction

Note: This section is currently under revision.

While the Kalman Filter is simple and elegant, it is not always obvious what it is doing conceptually or mathematically. Constructing your own with a guide, or using a code package with instructions is quite possible to do without understanding the filter. However, when choices must be made about the code, the hardware, or the values, or when general problems arise, a more thorough understanding becomes paramount.

The true algorithm for the Kalman filter is covered in the Kalman Filter section. This introduction will instead incrementally construct an equivalent algorithm starting from the concept of simple Linear Least Square Estimation, using only basic matrix operations and basic statistics.

Section 1.1 - Linear Least Squares Estimation

The Kalman Filter relies on a simple underlying concept – the linear least squares estimation. Given multiple noisy measurements of some state (speed, depth, acceleration, voltage, etc) the **LLSE** is an estimate that optimizes for the minimum of the sum of the squares of the errors.

In more formal terms, for some m measurements Y that are linear functions of a system with n unknown states X where $m>=n$. Such systems are said to be *over-determined,* whereby it is impossible to choose values of X that will satisfy every measurement perfectly, and thus a compromise of values of X is chosen that minimizes the total sum of the squares of the error between each measurement

$$ X_{est} = \text{arg}\,\min\limits_{\beta}\sum\|y-X\beta||^2 $$

Given the matrix format:

$$ \beta X = Y $$

$$ \begin{equation} \label{eq:control:expanded} \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \dots & \beta_{1,n} \\ \beta_{2,1} & \beta_{2,2} & \dots & \beta_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m,1} & \beta_{m,2} & \dots & \beta_{m,n} \end{bmatrix} \begin{bmatrix} X_{1} \\ X_{2} \\ \vdots \\ X_{n} \end{bmatrix} = \begin{bmatrix} Y_{1} \\ Y_{2} \\ \vdots \\ Y_{m} \\ \end{bmatrix} \end{equation} $$ This calculation could be performed iteratively, and the minimizing X_{est} discovered, but with the measurements enumerated in this format, matrix arithmetic offers us a simple way to solve for X_{est}. For those that recall their Geometry class in high school, to solve for X we need simply invert β and multiply that inverse by both sides.

$$ \beta^{-1} \beta X = \beta^{-1} Y \\ IX = \beta^{-1} Y \\ X = \beta^{-1} Y $$

However, you can only invert square matrices. For all $m \neq n$ this won't be the case. Here we employ the Moore-Penrose LLSE calculation.

First both sides are multiplied by the transpose of β.

$$\beta' \beta X = \beta' Y $$

Recall: $$ \beta' = \begin{bmatrix} \beta_{1,1} & \beta_{2,1} & \dots & \beta_{m,1} \\ \beta_{1,2} & \beta_{2,2} & \dots & \beta_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1,n} & \beta_{2,n} & \dots & \beta_{n,m} \end{bmatrix} $$

$\beta' \beta$ will be a square matrix of size n. Assuming that at least one measurement of all states X have been included, and indicated in β this new square matrix will be invertable. We can then multiply both sides by that inverse to isolate the X state vector.

$$ (\beta' \beta)^{-1}\beta' \beta X = (\beta' \beta)^{-1}\beta' Y \\ X = (\beta' \beta)^{-1}\beta' Y $$

Remarkably, this equation will give us an X vector that satisfies the LLSE optimization.

Example:

If we want to fit a line to two points, both data points must satisfy the line equation $y = mx + b$.

$$y_1 = mx_1 + b, \\ y_2 = mx_2+b $$

We can pose the mathematical question in a matrix-format:

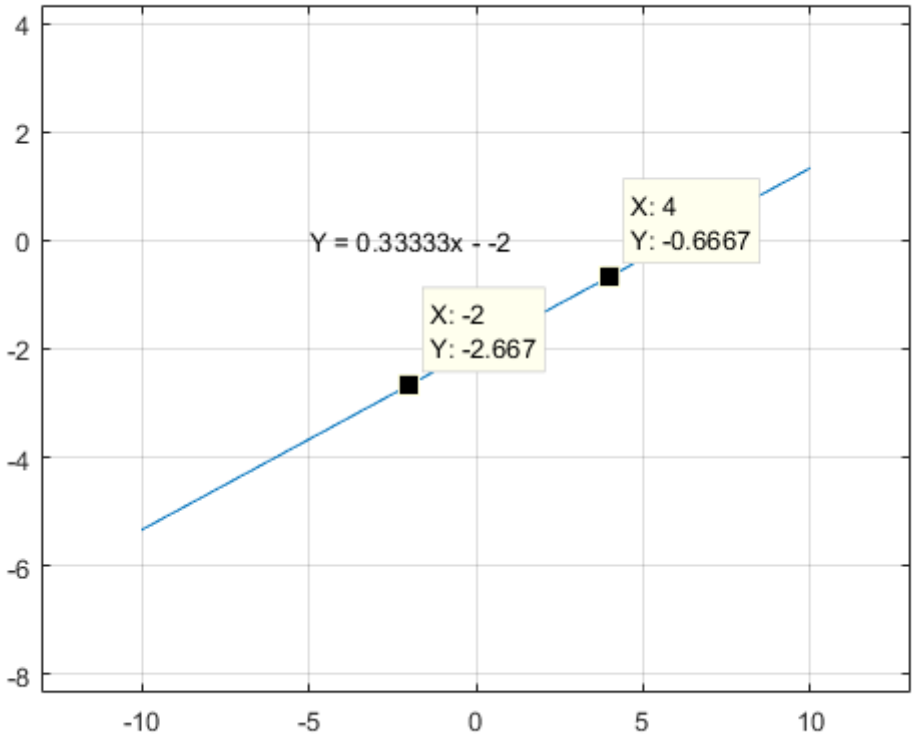
$$ \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} $$

Here our unknown slope and y-intercept m and b form our unknown state vector X. Our dependent measurements y are used to construct our measurement vector Y, and the linear combination of independent variables and constants that relate that measurement to our state form β.

Given the two coordinate pairs $(-2, -\frac{8}{3})$ and $(4, -\frac{2}{3})$ we can find the line that is defined by them.

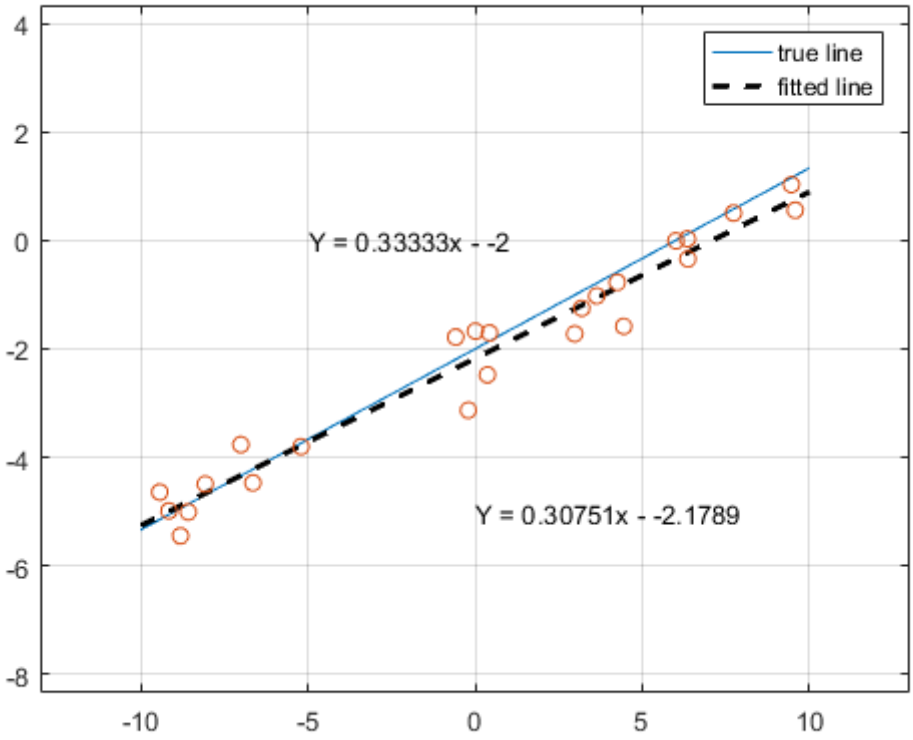
$$ \beta = \begin{bmatrix} -2 & 1 \\ 4 & 1 \end{bmatrix},\qquad X = \begin{bmatrix} m \\ b \end{bmatrix},\qquad Y = \begin{bmatrix} -\frac{8}{3} \\ -\frac{2}{3} \end{bmatrix} \\ \beta^{-1}Y = X \\ \begin{bmatrix} -2 & 1 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\frac{8}{3} \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -2 \end{bmatrix} $$

Thus our slope $m = \frac{1}{3}$ and our linear offset $b = -2$.



Now let's try fitting a line to an over-determined set of points. Below 20 points have been randomly generated. The x components were are uniformly random samples across domain $[-10, 10]$. The corresponding y components were first calculated directly using the true line equation $y = -\frac{1}{3}x -2$, and subsequently adding samples from a Gaussian random distribution with standard deviation $\sigma = 0.5$.

$$ \beta = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_{20} & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} m \\ b \end{bmatrix}, \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{20} \end{bmatrix} $$



We've just *estimated* our *state* based off of *noisy* measurements in an optimal fashion. At it's core, line-fitting like this is all that the Kalman Filter is doing. These next sections we will continuously build upon this basic function until we have something resembling the Kalman Filter.

Section 1.2 - Performance of Multiple Noisy Sensors

We can guess intuitively that the noisier our sensors, the worse our estimation. Likewise, the more sensors (measurements) we obtain, the better our estimation. Bayesian statistics tells us that all information, no matter how noisy, is still *good* information. Indeed the entire point of this field of mathematics is to get very accurate estimations from a combination of far-less accurate measurements. But exactly *how much* better do our estimates get?

Let's consider 4 depth sensors. Each depth sensor makes a noisy measurement z_n of the depth of our submarine.

For a quick refresher, a Guassian Random Variable has a mean and a variance. The Variance **(var)**of the distribution is equal to σ^2. Standard Deviation **(std)** is simply σ. **Std** and **var** are both perfectly valid ways to describe the distribution, and their usage depends mostly on ease of understanding, or ease of mathematical operations. As a good way to conceptualize what the **std** translates to, 68% of values pulled from a Gaussian distribution will be within $\pm1\sigma$ of the mean. 95% of the values will fall within $\pm2\sigma$.

If we simply read of one depth sensor, the estimate of our depth will have an equal noise.

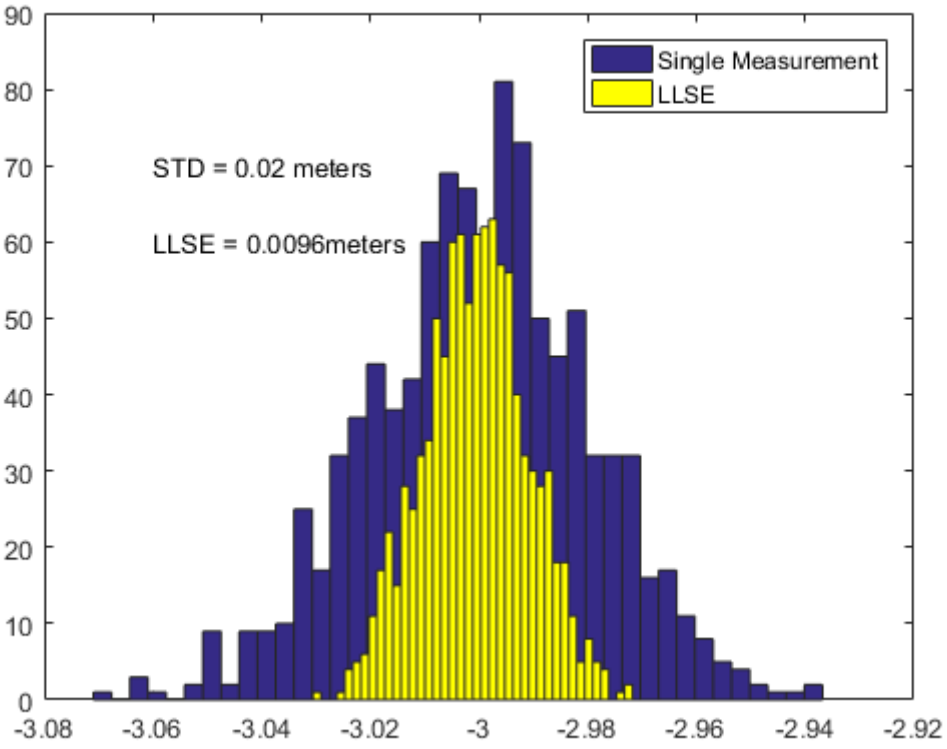
If we average the results of all four depth sensors, we will get a better estimate. Specifically, the Variance of an estimate is inversely proportional to the number of measurements n taken.

$$z_1 \sim\mathcal{N}(0,\sigma^2),\qquad \frac{z_1 + z_2 + z_3 + z_4}{4} \sim\mathcal{N}(0,\frac{\sigma^2}{4}),\qquad $$

and thus the estimate from n depth sensors of std σ will have an std of $\frac{\sigma}{\sqrt{n}}$. For this example, we'll assume the noise is Gaussian, with a mean of zero (no bias) and a standard deviation of $\sigma = 2cm = 0.02m$. The actual depth we're measuring is $3m$ Thus, we'd expect the distribution of our estimate to be $\sigma/2$ or $0.01m$. Let's see if that happens.

$$ \beta = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},\qquad X = \begin{bmatrix}Z\end{bmatrix},\qquad Y = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \\ X_{est} = (\beta'\beta)^{-1}\beta Y $$

If we create a script that generates 4 measurements by taking our true depth and adding a sample from a Guassian distribution with std $\sigma = 0.02$, then we can use the above equation to estimate our depth. $y_{1,2,3,4} = 3 + \sim\mathcal{N}(0,.02^2)$. If we estimate our depth 1000 times, we should get a distribution of estimated depths that has a standard deviation of $\frac{\sigma}{\sqrt{4}} = \frac{0.02m}{2} = 0.01m$.$



Now we know how confident we can be in our estimates given multiple, identical sensors.

Section 1.3 - Weighted Least Squares

The underlying assumption of the Linear Least Squares Estimation is that all measurements hold equal weight. That is to say, all are equally noisy and should be trusted equally. This can work well for fusing the results of duplicate sensors, but becomes a poor assumption when combining different

sensors.

Some of the cleverer readers might be thinking of a simple workaround – just add copies of the more accurate sensors' measurements into the matrix so the system listens to them more. And that'd certainly emulate what we're trying to do. But it's not mathematically perfect, nor is it elegant to code or computationally efficient to run.

If you have different sensors of different quality, we need to move to the *Weighted Linear Least Squares Estimation* **(WLLSE)**.

Before for **LLSE** we had:

$$ X = (\beta' \beta)^{-1}\beta' Y $$

Our new equation for *WLLSE* is:

$$ X = (\beta'W \beta)^{-1}\beta' WY,\qquad W = \begin{bmatrix} \frac{1}{\sigma_{1}^2} & 0 & \dots & 0 \\ 0 & \frac{1}{\sigma_{2}^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \frac{1}{\sigma_{n}^2} \end{bmatrix} $$

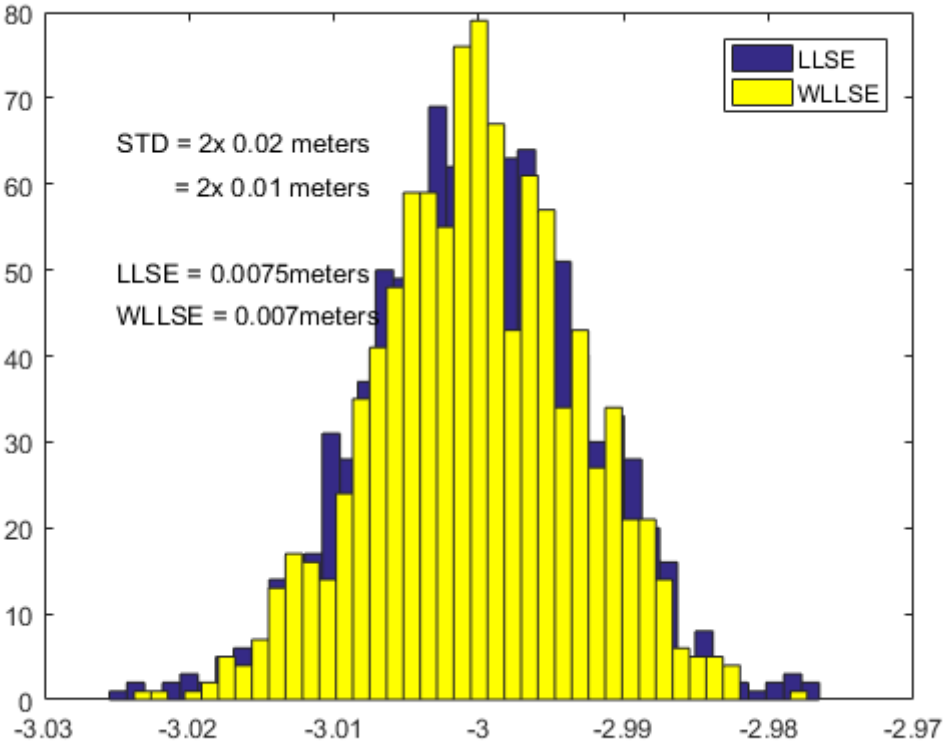
Working out the exact operations going on here is an exercise left to the student. However, intuitively we can see what's happening. When you multiply a matrix by an identity matrix I_n it remains unchanged. More generally, If you multiply a matrix by a diagonal matrix, it will scale each row or column (depending on order of operation) by it's corresponding value. Here we're inserting a scaling diagonal matrix in the middle of our **LLSE** solution. And we're scaling each value by the inverse of its variance.

Thus, measurements enumerated in the β matrix with low variances will be scaled by large numbers, and those with large variances will be similarly shrunk. In this way, the measurements are given more weight according to how well they can be trusted.

If we repeat our example from Section 1.2, but reduce the noise of two of the depth sensors, we should get an ever better estimation. For Depth sensors 1 and 2, $\sigma = 0.02m$. For Depth Sensors 3 and 4, $\sigma = 0.01$.

$$ \beta = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},\qquad X = \begin{bmatrix}Z\end{bmatrix},\qquad Y = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix},\qquad W = \begin{bmatrix} \frac{1}{0.02^2} & 0 & 0 & 0 \\ 0 & \frac{1}{0.02^2} & 0 & 0 \\ 0 & 0 & \frac{1}{0.01^2} & 0 \\ 0 & 0 & 0 & \frac{1}{0.01^2} \end{bmatrix} \\ X_{est} = (\beta'W\beta)^{-1}\beta WY $$

Recall that in **Section 1.2** we used four depth sensors of **std** $\sigma_z = 0.02$ and got an estimate with **std** $\sigma_{est} = 0.01m$.



By using a regular **LLSE**, where we assume all four depth sensors are equally trustworthy, we get a reduced std of $\sigma_{est} = 0.0075m$ simply by virtue of having more accuracy in some sensors. However, by using **WLLSE** we tighten the estimate down to $\sigma_{est} = 0.007$. Both estimates made here were made with identical information. The only question was how we processed that information. In this regard, appropriately trusting different sensors by different amounts gives us an overall better estimate.

However, the question now is, *exactly* how much should we trust our overall estimate? It's fine to make many estimates many times and find that distribution, but there should be *some* way to mathematically *know* how trustworthy our estimate is, based off of how trustworthy our various sensors were.

There is. In fact, we already calculated it. $$ R = inv(\beta' W \beta) = 5.413e^{-5},\qquad R = \begin{bmatrix} \sigma_{1}^2 & \sigma_{1}\sigma_{2} & \dots & \sigma_{1}\sigma_{n} \\ \sigma_{2}\sigma_{1} & \sigma_{2}^2 & \dots & \sigma_{2}\sigma_{n} \\ \vdots & \vdots & \ddots & \vdots & \\ \sigma_{n}\sigma_{1} & \sigma_{n}\sigma_{2} & \dots & \sigma_{n}^2 \end{bmatrix} \\ \sigma^2 = 5.413e^{-5},\qquad \sigma = 0.0756 $$

Where R is the co-variance matrix of the estimated state. The diagonals of R reflect the variance of each estimated state. The off-diagonal terms are co-variances. Put simply, two random variables can have normal distributions, but share some underlying property that makes them correlate. If one is above the mean, the other is likely above the mean. Or vice-versa. This is important when you start trying to calculate things from linear combinations of state elements. Mathematically, $$var(aX+bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X,Y)$$. Normally separate states will be uncorrelated, and have $Cov(X,Y)=0$. However, when you're dealing with physical models and sensors, correlations start to pop up, like the position and the velocity of your vehicle.

If you had an overestimated position but an underestimated velocity, and you try to predict where you'll be in 1 second, your predicted position could be pretty accurate. But if you know that more

often that not, your position and velocity are both over or under estimated, it's more likely the predicted position is *further* from the true position.

Thus, if you try to predict your location 1 second in the future, $x[k+1] = x[k] + \dot{x}[k] * 1sec$, the variance of that predicted location isn't going to simply be the noise of your current position plus the noise of your velocity, but additionally some extra noise because both states are inter-related and more liable to compound their noise rather than counteract it. That's where the $2cov(X,Y)$ term comes in.

And if you think that equation for variance looks remarkably like the Law of Cosines in Trigonometry; Congratulations. You get a cookie.